WS#10-3
Properties of Determinants

Determinant of A 2x2 matrix $\rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Determinants have several properties that are helpful in quickly determining their value. Some are listed below:

1. The value of a determinant changes sign if any two rows (or any two columns) are interchanged.

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3(2) - 4(1) = \boxed{2} \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3) = \boxed{-2}$$

2. If all the entries in a row (or any column) equal 0, the value of the determinant is 0.

3. If any two rows (or any two columns) of a determinant have corresponding entries that are equal, the value of the determinant is 0.

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

b/c row 1 + row 2 are identical.

4. If any row (or any column) of a determinant is multiplied by a non-zero number k, the value of the determinant is also changed by a factor of k.

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3(2) - 4(1) = \boxed{2} \qquad \begin{vmatrix} 3k & 4k \\ 1 & 2 \end{vmatrix} = 3k(2) - 4k(1) = 6k - 4k = \boxed{2k}$$

5. If the entries of any row (or any column) of a determinant are multiplied by a nonzero number k and the result is added to the corresponding entries of another row (or column), the value of the determinant remains unchanged.

$$\begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} = 3(2) - 4(5) = \boxed{-14} \qquad \begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} R_1 \rightarrow -2r_2 + r_1 = \begin{vmatrix} -7 & 0 \\ 5 & 2 \end{vmatrix} = -7(2) - 0(5)$$

$$= \boxed{-14}$$

Cramer's Rule For two equations containing two variables

$$\left.\begin{aligned} ax+by&=s \\ cx+dy&=t \end{aligned}\right\}$$ The solution to this system is given by:

$$x = \frac{\begin{vmatrix} s & b \\ t & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

→ Provided that $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$$

- In the solution for y, the numerator (D_y) is formed by replacing entries in the 2nd column (the coef. of y) of D by the constants on the (R) side of the equal sign. $D_y = \begin{vmatrix} a & s \\ c & t \end{vmatrix}$

- In the solution for x, the numerator (D_x) is formed by replacing entries in the first column (the coefficients of x) of D by the constants on the (R) side of the equal sign. $D_x = \begin{vmatrix} s & b \\ t & d \end{vmatrix}$

Ex. 1 Solve the following system using Cramer's Rule if applicable

$$\begin{aligned} 3x - 2y &= 4 \\ 6x + y &= 13 \end{aligned}$$

$D = \begin{vmatrix} 3 & -2 \\ 6 & 1 \end{vmatrix} = 3(1) - (-2)(6) = \underline{15 \neq 0}$, so we can use Cramer's Rule.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 4 & -2 \\ 13 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 6 & 1 \end{vmatrix}} = \frac{4(1) - (-2)(13)}{3(1) - (-2)(6)} = \frac{30}{15} = \boxed{2}, \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} 3 & 4 \\ 6 & 13 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 6 & 1 \end{vmatrix}} = \frac{3(13) - 4(6)}{3(1) - (-2)(6)} = \frac{15}{15} = \boxed{1}$$

Solution: $\boxed{x = 2,\ y = 1 \text{ or } (2, 1)}$

- For an $n \times n$ determinant A, the cofactor of entry a_{ij}, denoted by A_{ij}, is: $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of entry a_{ij}.

Ex. 2 Evaluate $\begin{vmatrix} 3 & 4 & -1 \\ 4 & 6 & 2 \\ 8 & -2 & 3 \end{vmatrix}$

- You may expand across any row or column to find the determinant

Solution: I will expand across row 1

$$(-1)^{1+1} \cdot 3 \begin{vmatrix} 6 & 2 \\ -2 & 3 \end{vmatrix} + (-1)^{1+2} \cdot 4 \begin{vmatrix} 4 & 2 \\ 8 & 3 \end{vmatrix} + (-1)^{1+3} \cdot (-1) \begin{vmatrix} 4 & 6 \\ 8 & -2 \end{vmatrix}$$

$$(1)(3)[6(3) - 2(-2)] + (-1)(4)[4(3) - 2(8)] + (1)(-1)[4(-2) - 6(8)]$$

$$= 3(22) - 4(-4) - 1(-56) = 66 + 16 + 56 = \boxed{138}$$

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ Expand across row 1 to solve.

$$= (-1)^{1+1} \cdot (a_{11}) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} \cdot (a_{12}) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} \cdot (a_{13}) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cramer's Rule for 3 equ. containing 3 variables:

$$\left\{\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= c_1 \\ a_{21}x + a_{22}y + a_{23}z &= c_2 \\ a_{31}x + a_{32}y + a_{33}z &= c_3 \end{aligned}\right.$$

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D} \quad \text{if } D \neq 0$$

where $D_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}$, $D_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}$, $D_z = \begin{vmatrix} a_{11} & a_{12} & c_{13} \\ a_{21} & a_{22} & c_{23} \\ a_{31} & a_{32} & c_{33} \end{vmatrix}$, $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$